

SYMMETRIES AND CONSERVATION LAWS IN THE GÜNTHER k -SYMPLECTIC FORMALISM OF FIELD THEORY

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Abstract

This paper is devoted to studying symmetries of k -symplectic Hamiltonian and Lagrangian first-order classical field theories. In particular, we define symmetries and Cartan symmetries and study the problem of associating conservation laws to these symmetries, stating and proving Noether's theorem in different situations for the Hamiltonian and Lagrangian cases. We also characterize equivalent Lagrangians, which lead to an introduction of Lagrangian gauge symmetries, as well as analyzing their relation with Cartan symmetries.

Key words: *Symmetries, Conservation laws, Noether theorem, Lagrangian and Hamiltonian field theories, k -symplectic manifolds.*

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1 Introduction

Günther's paper [21] gives a geometric Hamiltonian formalism for field theories. The crucial device is the introduction of a vector-valued generalization of a symplectic form, called a polysymplectic form. One of the advantages of this formalism is that one only needs the tangent and cotangent bundle of a manifold to develop it. In [37] Günther's formalism has been revised and clarified. It has been shown that the polysymplectic structures used by Günther to develop his formalism could be replaced by the k -symplectic structures defined by Awane [3, 5]. So this formalism is also called k -symplectic formalism.

The k -symplectic formalism is the generalization to field theories of the standard symplectic formalism in Mechanics, which is the geometric framework for describing autonomous dynamical systems. In this sense, the k -symplectic formalism is used to give a geometric description of certain kinds of field theories: in a local description, those theories whose Lagrangian does not depend on the base coordinates, denoted by (t^1, \dots, t^k) (in many of these, the space-time coordinates); that is, the k -symplectic formalism is only valid for Lagrangians $L(q^i, v_A^i)$ and

Hamiltonians $H(q^i, p_i^A)$ that depend on the field coordinates q^i and on the partial derivatives of the field v_A^i , or the corresponding moment p_i^A . A natural extension of this formalism is the so-called k -cosymplectic formalism, which is the generalization to field theories of the cosymplectic formalism geometrically describing non-autonomous mechanical systems (this description can be found in [32, 33]). This formalism is devoted to describing field theories involving the coordinates (t^1, \dots, t^k) on the Lagrangian $L(t^A, q^i, v_A^i)$ and on the Hamiltonian $H(t^A, q^i, p_i^A)$.

Let us remark here that the polysymplectic formalism developed by Sardanashvily [13], based on a vector-valued form defined on some associated fiber bundle, is a different description of classical field theories of first order than the polysymplectic (or k -symplectic) formalism proposed by Günther (see also [22] for more details). We must also remark that the soldering form on the linear frames bundle is a polysymplectic form, and its study and applications to field theory, constitute the n -symplectic geometry developed by L. K. Norris in [39, 40, 41, 42, 43].

An alternative way to derive the field equations is to use the so-called multisymplectic formalism, developed by Tulczyjew's school in Warsaw (see [23, 24, 25, 48]), and independently by García and Pérez-Rendón [11, 12] and Goldschmidt and Sternberg [14]. This approach was revised by Martin [35, 36] and Gotay et al [15, 16, 17, 18] and more recently by Cantrijn *et al* [7, 8].

The aim of this paper is to study symmetries and conservation laws on first-order classical field theories, both for the Lagrangian and Hamiltonian formalisms, using Günther's k -symplectic description, and considering only the regular case. The study of symmetries of k -symplectic Hamiltonian systems, is, of course, a topic of great interest. The general problem of a group of symmetries acting on a k -symplectic manifold and the subsequent theory of reduction has recently been analyzed in [37]. Here, we recover the idea of *conservation law* or *conserved quantity*, and state Noether's theorem for Hamiltonian and Lagrangian systems in k -symplectic field theories. Thus, a large part of our discussion is a generalization of the results obtained for non-autonomous mechanical systems (see, in particular, [27], and references quoted therein). We further remark that the problem of symmetries in field theory has also been analyzed using other geometric frameworks, such as the *multisymplectic models* (see, for instance, [10, 18, 28]).

The organization of the paper is as follows: Sections 2 and 3 are devoted to the study of symmetries and conservation laws in Hamiltonian k -symplectic field theory and Lagrangian k -symplectic field theory, respectively. In particular, in Sections 2.1 and 2.2 we develop the Hamiltonian formalism. So, in Section 2.1 the field theoretic phase space of moments is introduced as the Whitney sum $(T_k^1)^*Q$ of k -copies of the cotangent bundle T^*Q of a manifold Q . This space is the canonical example of polysymplectic manifold introduced by Günther and k -symplectic manifolds introduced by Awane [3, 4, 5]. In Section 2.2, the Hamiltonian k -symplectic formalism is described. In Section 2.3 we obtain the main results of this Section: after introducing different kinds of symmetries and their relation, we can associate to some of them (the so-called *Cartan symmetries*) a conservation law (*Noether's Theorem*).

Concerning the Lagrangian formalism (Section 3), the field theoretic state space of velocities is introduced in Section 3.1 as the Whitney sum T_k^1Q of k -copies of the tangent bundle TQ of a manifold Q . This manifold has a canonical k -tangent structure defined by k tensors fields of type $(1, 1)$. The k -tangent manifolds were introduced in de León *et al.* [29, 30], and they generalize the tangent manifolds. A geometric interpretation of the second order partial differential equations is also given. Here we show that these equations can be characterized using the canonical k -tangent structure of T_k^1Q , which generalizes the case of Classical Mechanics. The Lagrangian formalism is developed in Section 3.2, where the canonical k -tangent structure of T_k^1Q is used for its construction instead of the Legendre transformation as in Günther [21]. In Section 3.3 we discuss symmetries and conservation laws in the Lagrangian case, obtaining results analogous

to those in Section 2.3, including the corresponding Noether's theorem. Finally, in Sections 3.4 and 3.5 we introduce the notion of gauge equivalent Lagrangians, showing that they give the same solutions to the Euler-Lagrange equations. This leads to the introduction of the so-called *Lagrangian gauge symmetries*, and to stating a version of the Noether theorem for a particular class of them.

All manifolds are real, paracompact, connected and C^∞ . All maps are C^∞ . Sum over crossed repeated indices is understood.

2 Hamiltonian k -symplectic case

2.1 Geometric elements

2.1.1 The cotangent bundle of k^1 -covelocities of a manifold. Canonical structures

Let Q be a differentiable manifold of dimension n and $\tau_Q^*: T^*Q \rightarrow Q$ its cotangent bundle. We denote by $(T_k^1)^*Q = T^*Q \oplus \dots \oplus T^*Q$ the Whitney sum of k copies of T^*Q , with projection map $\tau^*: (T_k^1)^*Q \rightarrow Q$, $\tau^*(\alpha_{1_q}, \dots, \alpha_{k_q}) = q$.

The manifold $(T_k^1)^*Q$ can be canonically identified with the vector bundle $J^1(Q, \mathbb{R}^k)_0$ of k^1 -covelocities of the manifold Q , the manifold of 1-jets of maps $\sigma: Q \rightarrow \mathbb{R}^k$ with target at $0 \in \mathbb{R}^k$ and projection map $\tau^*: J^1(Q, \mathbb{R}^k)_0 \rightarrow Q$, $\tau^*(j_{q,0}^1\sigma) = q$; that is,

$$\begin{aligned} J^1(Q, \mathbb{R}^k)_0 &\equiv T^*Q \oplus \dots \oplus T^*Q \\ j_{q,0}^1\sigma &\equiv (d\sigma^1(q), \dots, d\sigma^k(q)) \end{aligned}$$

where $\sigma^A = \pi_A \circ \sigma: Q \rightarrow \mathbb{R}$ is the A -th component of σ , and $\pi_A: \mathbb{R}^k \rightarrow \mathbb{R}$ are the canonical projections, $1 \leq A \leq k$. For this reason, $(T_k^1)^*Q$ is also called *the bundle of k^1 covelocities of the manifold Q* .

If (q^i) are local coordinates on $U \subseteq Q$, then the induced local coordinates (q^i, p_i) , $1 \leq i \leq n$, on $T^*U = (\tau_Q^*)^{-1}(U)$, are given by

$$q^i(\alpha_q) = q^i(q), \quad p_i(\alpha_q) = \alpha_q \left(\frac{\partial}{\partial q^i} \Big|_q \right)$$

and the induced local coordinates (q^i, p_i^A) , $1 \leq i \leq n$, $1 \leq A \leq k$, on $(T_k^1)^*U = (\tau^*)^{-1}(U)$ are

$$q^i(\alpha_{1_q}, \dots, \alpha_{k_q}) = q^i(q), \quad p_i^A(\alpha_{1_q}, \dots, \alpha_{k_q}) = \alpha_{A_q} \left(\frac{\partial}{\partial q^i} \Big|_q \right).$$

If $\tau_Q^*: T^*Q \rightarrow Q$ is the canonical projection, the Liouville 1-form $\theta \in \Omega^1(T^*Q)$ is given by

$$\theta(\alpha_q)(\tilde{X}_{\alpha_q}) = \alpha_q((\tau_Q^*)_*(\alpha_q)(\tilde{X}_{\alpha_q})), \quad \alpha_q \in T^*Q, \quad \tilde{X}_{\alpha_q} \in T_{\alpha_q}(T^*Q),$$

then $\omega = -d\theta$ is the canonical symplectic structure in T^*Q , and therefore we define

$$\omega^A = (\tau_A^*)^*\omega, \quad 1 \leq A \leq k,$$

where $\tau_A^*: (T_k^1)^*Q \rightarrow T^*Q$ is the canonical projection on the A^{th} -copy T^*Q of $(T_k^1)^*Q$. Of course, $\omega^A = -d\theta^A$, where $\theta^A = (\tau_A^*)^*\theta$. Thus, the *canonical k -symplectic structure* on $(T_k^1)^*Q$ is given by the family $(\omega^A, V; 1 \leq A \leq k)$, in $(T_k^1)^*Q$, where $V = \ker(\tau^*)_*$ (see [3, 5, 37]).

As the canonical symplectic structure on T^*Q is locally given by $\omega = -d(p_i dq^i) = dq^i \wedge dp_i$, then the canonical forms ω^A in $(T_k^1)^*Q$ are locally given by

$$\omega^A = -d\theta^A = -d(p_i^A dq^i) = dq^i \wedge dp_i^A. \quad (1)$$

It is interesting to recall that the *canonical polysymplectic structure* in $(T_k^1)^*Q$ introduced by Günther [21] is the closed non-degenerate \mathbb{R}^k -valued 2-form $\bar{\omega} = \omega^A \otimes r_A$, where $\{r_1, \dots, r_k\}$ denotes the canonical basis of \mathbb{R}^k .

2.1.2 Complete lift of diffeomorphisms and vector fields from Q to $(T_k^1)^*Q$

Now, let $\varphi: Q \rightarrow Q$ be a diffeomorphism, then the cotangent map $T^*\varphi: T^*Q \rightarrow T^*Q$ is given by $T^*\varphi(\alpha_q) = \alpha_q \circ \varphi_*(\varphi^{-1}(q))$. We define the *canonical prolongation* of φ to $(T_k^1)^*Q$ as the map $(T_k^1)^*\varphi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$ given by

$$(T_k^1)^*\varphi(\alpha_{1q}, \dots, \alpha_{kq}) = (T^*\varphi(\alpha_{1q}), \dots, T^*\varphi(\alpha_{kq})) \quad , \quad \text{for } (\alpha_{1q}, \dots, \alpha_{kq}) \in (T_k^1)^*Q, q \in Q.$$

If Z is a vector field on Q , with local 1-parametric group of transformations $h_s: Q \rightarrow Q$ then the local 1-parametric group of transformations $(T_k^1)^*(h_s): (T_k^1)^*Q \rightarrow (T_k^1)^*Q$ generates a vector field Z^{C*} on $(T_k^1)^*Q$, which is called the *canonical lift* of Z to $(T_k^1)^*Q$. If $Z = Z^i \frac{\partial}{\partial q^i}$, the local expression of Z^{C*} is

$$Z^{C*} = Z^i \frac{\partial}{\partial q^i} - p_j^A \frac{\partial Z^j}{\partial q^k} \frac{\partial}{\partial p_k^A}.$$

The canonical liftings or prolongations of diffeomorphisms and vector fields on the base manifold Q to $(T_k^1)^*Q$ have the following properties:

Lemma 1 *1. Let $\varphi: Q \rightarrow Q$ be a diffeomorphism and let $\Phi = (T_k^1)^*\varphi$ be the canonical prolongation of φ to $(T_k^1)^*Q$. Then:*

$$(i) \Phi^*\theta^A = \theta^A \quad , \quad (ii) \Phi^*\omega^A = \omega^A.$$

2. Let $Z \in \mathfrak{X}(Q)$, and let Z^{C} be the canonical prolongation of Z to $(T_k^1)^*Q$. Then*

$$(i) L(Z^{C*})\theta^A = 0 \quad , \quad (ii) L(Z^{C*})\omega^A = 0. \quad (2)$$

(Proof)

1. Part (i) is a consequence of the commutation rule $\tau_A^* \circ (T_k^1)^*\varphi = T^*\varphi \circ \tau_A^*$. In fact,

$$\begin{aligned} [(T_k^1)^*\varphi]^* \theta^A &= [(T_k^1)^*\varphi]^* ((\tau_A^*)^* \theta) = [(\tau_A^*)^* \circ (T_k^1)^*\varphi]^* \theta = (T^*\varphi \circ \tau_A^*)^* \theta \\ &= (\tau_A^*)^* ((T^*\varphi)^* \theta) = (\tau_A^*)^* \theta = \theta^A, \end{aligned}$$

where we have used that $(T^*\varphi)^* \theta = \theta$ (see [1], pag. 180).

Part (ii) is a direct consequence of (i).

2. Since the infinitesimal generator of the complete lift Z^{C*} of Z is the canonical prolongation of the infinitesimal generator of Z , from the first item we conclude that (2) holds.

■

2.1.3 k -vector fields

Let M be a differentiable manifold. Denote by $T_k^1 M$ the Whitney sum $TM \oplus \dots \oplus TM$ of k copies of TM , with projection $\tau : T_k^1 M \rightarrow M$, $\tau(v_{1q}, \dots, v_{kq}) = q$.

Definition 1 A k -vector field on M is a section $\mathbf{X} : M \rightarrow T_k^1 M$ of the projection τ .

Since $T_k^1 M$ is the Whitney sum $TM \oplus \dots \oplus TM$ of k copies of TM , we deduce that a k -vector field \mathbf{X} defines a family of k vector fields $X_1, \dots, X_k \in \mathfrak{X}(M)$ by projecting \mathbf{X} onto every factor; that is, $X_A = \tau_A \circ \mathbf{X}$, where $\tau_A : T_k^1 M \rightarrow TM$ is the canonical projection on the A^{th} -copy TM of $T_k^1 M$.

Definition 2 An integral section of the k -vector field $\mathbf{X} = (X_1, \dots, X_k)$, passing through a point $q \in M$, is a map $\psi : U_0 \subset \mathbb{R}^k \rightarrow M$, defined on some neighborhood U_0 of $0 \in \mathbb{R}^k$, such that

$$\psi(0) = q, \quad \psi_*(t) \left(\frac{\partial}{\partial t^A} \Big|_t \right) = X_A(\psi(t)) \quad , \quad \text{for every } t \in U_0, 1 \leq A \leq k$$

or, what is equivalent, ψ satisfies that $X \circ \psi = \psi^{(1)}$, where $\psi^{(1)}$ is the first prolongation of ψ to $T_k^1 M$ defined by

$$\begin{aligned} \psi^{(1)} : U_0 \subset \mathbb{R}^k &\longrightarrow T_k^1 M \\ t &\longrightarrow \psi^{(1)}(t) = j_0^1 \psi_t \equiv \left(\psi_*(t) \left(\frac{\partial}{\partial t^1} \Big|_t \right), \dots, \psi_*(t) \left(\frac{\partial}{\partial t^k} \Big|_t \right) \right) . \end{aligned}$$

A k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ on M is integrable if there is an integral section passing through every point of M .

In local coordinates, we have

$$\psi^{(1)}(t^1, \dots, t^k) = \left(\psi^i(t^1, \dots, t^k), \frac{\partial \psi^i}{\partial t^A}(t^1, \dots, t^k) \right), \quad 1 \leq A \leq k, 1 \leq i \leq n. \quad (3)$$

2.2 Hamiltonian formalism: k -symplectic Hamiltonian systems

Let $H : (T_k^1)^* Q \rightarrow \mathbb{R}$ be a Hamiltonian function. The family $((T_k^1)^* Q, \omega^A, H)$ is called a k -symplectic Hamiltonian system. The Hamilton-de Donder-Weyl equations for this system are the following set of partial differential equations

$$\frac{\partial H}{\partial q^i} \Big|_{\psi(t)} = - \sum_{A=1}^k \frac{\partial \psi_i^A}{\partial t^A} \Big|_t, \quad \frac{\partial H}{\partial p_i^A} \Big|_{\psi(t)} = \frac{\partial \psi^i}{\partial t^A} \Big|_t, \quad 1 \leq i \leq n, 1 \leq A \leq k, \quad (4)$$

where $\psi : \mathbb{R}^k \rightarrow (T_k^1)^* Q$, $\psi(t) = (\psi^i(t), \psi_i^A(t))$, is a solution.

We denote by $\mathfrak{X}_H^k((T_k^1)^* Q)$ the set of k -vector fields $\mathbf{X} = (X_1, \dots, X_k)$ on $(T_k^1)^* Q$, which are solutions to the equations

$$\sum_{A=1}^k i(X_A) \omega^A = dH. \quad (5)$$

Then, if $\mathbf{X} \in \mathfrak{X}_H^k((T_k^1)^* Q)$ is integrable, and $\psi : \mathbb{R}^k \rightarrow (T_k^1)^* Q$ is an integral section of \mathbf{X} , then, from (1), we obtain that $\psi(t) = (\psi^i(t), \psi_i^A(t))$ is a solution to the system (4).

2.3 Symmetries and conservation laws

Let $((T_k^1)^*Q, \omega^A, H)$ be a k -symplectic Hamiltonian system, and its associated Hamilton-de Donder-Weyl equations (4).

First, following [44], we introduce the next definition :

Definition 3 A conservation law (or a conserved quantity) for the Hamilton-de Donder-Weyl equations (4) is a map $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k): (T_k^1)^*Q \rightarrow \mathbb{R}^k$ such that the divergence of

$$\mathcal{F} \circ \psi = (\mathcal{F}^1 \circ \psi, \dots, \mathcal{F}^k \circ \psi): U_0 \subset \mathbb{R}^k \rightarrow \mathbb{R}^k$$

is zero for every solution ψ to the Hamilton-de Donder-Weyl equations (4); that is

$$\sum_{A=1}^k \frac{\partial(\mathcal{F}^A \circ \psi)}{\partial t^A} = 0.$$

Proposition 1 If $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k): (T_k^1)^*Q \rightarrow \mathbb{R}^k$ is a conservation law then for every integrable k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ in $\mathfrak{X}_H((T_k^1)^*Q)$, we have that

$$\sum_{A=1}^k L(X_A) \mathcal{F}^A = 0.$$

(Proof) If $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$ is integrable and $\psi: \mathbb{R}^k \rightarrow (T_k^1)^*Q$ is an integral section of \mathbf{X} , then the following relation holds for every $t \in \mathbb{R}^k$ and $A = 1, \dots, k$,

$$X_A(\psi(t)) = \psi_*(t) \left(\frac{\partial}{\partial t^A} \Big|_t \right)$$

and therefore

$$\sum_{A=1}^k L(X_A) \mathcal{F}^A = \sum_{A=1}^k \psi_*(t) \left(\frac{\partial}{\partial t^A} \Big|_t \right) (\mathcal{F}^A) = \sum_{A=1}^k \frac{\partial(\mathcal{F}^A \circ \psi)}{\partial t^A} \Big|_t = 0$$

since ψ is a solution to the Hamilton-de Donder-Weyl equations (4). ■

Remark: The case $k = 1$ corresponds to Classical Mechanics. In this case we know that \mathcal{F} is a constant of the motion if and only if $L(X_H)\mathcal{F} = 0$, where X_H is the Hamiltonian vector field defined by $i(X_H)\omega = dH$.

Definition 4 1. A symmetry of the k -symplectic Hamiltonian system $((T_k^1)^*Q, \omega^A, H)$ is a diffeomorphism $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$ such that, for every solution ψ to the Hamilton-de Donder-Weyl equations (4), we have that $\Phi \circ \psi$ is also a solution to these equations.

In the particular case that $\Phi = (T_k^1)^*\varphi$ for some $\varphi: Q \rightarrow Q$ (i.e.; Φ is the canonical lifting of some diffeomorphism in Q), the symmetry Φ is said to be natural.

2. An infinitesimal symmetry of the k -symplectic Hamiltonian system $((T_k^1)^*Q, \omega^A, H)$ is a vector field $Y \in \mathfrak{X}((T_k^1)^*Q)$ whose local flows are local symmetries.

In the particular case where $Y = Z^{C*}$ for some $Z \in \mathfrak{X}(Q)$, (i.e.; Y is the canonical lifting of some vector field in Q), the infinitesimal symmetry Y is said to be natural.

As a consequence of the definition, all the results that we state for symmetries also hold for infinitesimal symmetries.

A first straightforward consequence of definitions 3 and 4 is:

Proposition 2 *If $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$ is a symmetry of a k -symplectic Hamiltonian system and $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k): (T_k^1)^*Q \rightarrow \mathbb{R}^k$ is a conservation law, then so is $\Phi^*\mathcal{F} = (\Phi^*\mathcal{F}^1, \dots, \Phi^*\mathcal{F}^k)$.*

There is a class of symmetries which play a relevant role as generators of conserved quantities:

Proposition 3 *Let $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$ be a diffeomorphism. If*

$$\Phi^*\omega^A = \omega^A, \quad 1 \leq A \leq k \quad \text{and} \quad \Phi^*H = H \quad (\text{up to a constant}).$$

*then Φ is a symmetry of the k -symplectic Hamiltonian system $((T_k^1)^*Q, \omega^A, H)$.*

(Proof) We must prove that, if $\psi: U_0 \subset \mathbb{R}^k \rightarrow (T_k^1)^*Q$ is a solution to the Hamilton-de Donder-Weyl equations (4), then $\Phi \circ \psi$ is also a solution, that is,

$$(a) \quad \left. \frac{\partial H}{\partial q^i} \right|_{(\Phi \circ \psi)(t)} = - \sum_{A=1}^k \left. \frac{\partial(\Phi \circ \psi)_i^A}{\partial t^A} \right|_t, \quad (b) \quad \left. \frac{\partial H}{\partial p_i^A} \right|_{(\Phi \circ \psi)(t)} = \left. \frac{\partial(\Phi \circ \psi)^i}{\partial t^A} \right|_t.$$

In local coordinates, we write the diffeomorphism $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$ as follows

$$\Phi(q^j, p_j^B) = (\Phi^i(q^j, p_j^B), \Phi_i^A(q^j, p_j^B)) \quad .$$

The condition $\Phi^*\omega^A = \omega^A$ implies

$$\begin{aligned} 0 &= \left. \frac{\partial \Phi^i}{\partial q^j} \right|_w \left. \frac{\partial \Phi_i^A}{\partial q^k} \right|_w, \\ 0 &= \left. \frac{\partial \Phi^i}{\partial p_j^B} \right|_w \left. \frac{\partial \Phi_i^A}{\partial p_k^C} \right|_w, \\ \delta_j^k \delta_C^A &= \left. \frac{\partial \Phi^i}{\partial q^j} \right|_w \left. \frac{\partial \Phi_i^A}{\partial p_k^C} \right|_w - \left. \frac{\partial \Phi^i}{\partial p_k^C} \right|_w \left. \frac{\partial \Phi_i^A}{\partial q^j} \right|_w. \end{aligned} \tag{6}$$

Furthermore, since Φ is a diffeomorphism, $\Phi \circ \Phi^{-1} = Id_{(T_k^1)^*Q}$. Applying the chain rule we obtain:

$$\delta_k^i = \left. \frac{\partial(\Phi \circ \Phi^{-1})^i}{\partial q^k} \right|_w = \left. \frac{\partial \Phi^i}{\partial q^j} \right|_{\Phi^{-1}(w)} \left. \frac{\partial(\Phi^{-1})^j}{\partial q^k} \right|_w + \left. \frac{\partial \Phi^i}{\partial p_j^A} \right|_{\Phi^{-1}(w)} \left. \frac{\partial(\Phi^{-1})_j^A}{\partial q^k} \right|_w, \tag{7}$$

$$0 = \left. \frac{\partial(\Phi \circ \Phi^{-1})^i}{\partial p_k^B} \right|_w = \left. \frac{\partial \Phi^i}{\partial q^j} \right|_{\Phi^{-1}(w)} \left. \frac{\partial(\Phi^{-1})^j}{\partial p_k^B} \right|_w + \left. \frac{\partial \Phi^i}{\partial p_j^A} \right|_{\Phi^{-1}(w)} \left. \frac{\partial(\Phi^{-1})_j^A}{\partial p_k^B} \right|_w,$$

$$0 = \left. \frac{\partial(\Phi \circ \Phi^{-1})_i^A}{\partial q^j} \right|_w = \left. \frac{\partial \Phi_i^A}{\partial q^k} \right|_{\Phi^{-1}(w)} \left. \frac{\partial(\Phi^{-1})^k}{\partial q^j} \right|_w + \left. \frac{\partial \Phi_i^A}{\partial p_k^B} \right|_{\Phi^{-1}(w)} \left. \frac{\partial(\Phi^{-1})_k^B}{\partial q^j} \right|_w, \tag{8}$$

$$\delta_j^i \delta_C^A = \left. \frac{\partial(\Phi \circ \Phi^{-1})_i^A}{\partial p_j^C} \right|_w = \left. \frac{\partial \Phi_i^A}{\partial q^k} \right|_{\Phi^{-1}(w)} \left. \frac{\partial(\Phi^{-1})^k}{\partial p_j^C} \right|_w + \left. \frac{\partial \Phi_i^A}{\partial p_k^B} \right|_{\Phi^{-1}(w)} \left. \frac{\partial(\Phi^{-1})_k^B}{\partial p_j^C} \right|_w. \tag{9}$$

From the equations (6-9) we obtain

$$\left. \frac{\partial \Phi^s}{\partial q^j} \right|_{\Phi^{-1}(w)} = \delta_B^A \left. \frac{\partial (\Phi^{-1})_j^A}{\partial p_s^B} \right|_w, \quad \delta_D^A \left. \frac{\partial \Phi^s}{\partial p_k^C} \right|_{\Phi^{-1}(w)} = -\delta_C^A \left. \frac{\partial (\Phi^{-1})^k}{\partial p_s^D} \right|_w \quad (10)$$

$$\left. \frac{\partial \Phi_s^A}{\partial q^j} \right|_{\Phi^{-1}(w)} = - \left. \frac{\partial (\Phi^{-1})_j^A}{\partial q^s} \right|_w, \quad \left. \frac{\partial \Phi_s^A}{\partial p_k^C} \right|_{\Phi^{-1}(w)} = \delta_C^A \left. \frac{\partial (\Phi^{-1})^k}{\partial q^s} \right|_w. \quad (11)$$

From the condition $\Phi^*H = H$ written as follows

$$H(q^j, p_j^B) = (H \circ \Phi)(q^j, p_j^B) = H(\Phi^i(q^j, p_j^B), \Phi_i^A(q^j, p_j^B)),$$

we obtain, for every $w \in (T_k^1)^*Q$.

$$\begin{aligned} \left. \frac{\partial H}{\partial q^j} \right|_w &= \left. \frac{\partial H}{\partial q^i} \right|_{\Phi(w)} \left. \frac{\partial \Phi^i}{\partial q^j} \right|_w + \left. \frac{\partial H}{\partial p_i^A} \right|_{\Phi(w)} \left. \frac{\partial \Phi_i^A}{\partial q^j} \right|_w \\ \left. \frac{\partial H}{\partial p_j^A} \right|_w &= \left. \frac{\partial H}{\partial q^i} \right|_{\Phi(w)} \left. \frac{\partial \Phi^i}{\partial p_j^A} \right|_w + \left. \frac{\partial H}{\partial p_i^B} \right|_{\Phi(w)} \left. \frac{\partial \Phi_i^B}{\partial p_j^A} \right|_w, \end{aligned} \quad (12)$$

Applying the chain rule, by a straightforward computation one proves (a) as consequence of (4), (6), (7), (8), (11) and (12), and taking into account (4), (10), (11) and (12), one proves (b). ■

The case $k = 1$ corresponds to Classical Mechanics. In this case the above result can be found in [34].

Taking into account this proposition, we introduce the following definitions:

Definition 5 1. A Cartan (or Noether) symmetry of a k -symplectic Hamiltonian system $((T_k^1)^*Q, \omega^A, H)$ is a diffeomorphism $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$ such that,

- (a) $\Phi^*\omega^A = \omega^A$, for $A = 1, \dots, k$.
- (b) $\Phi^*H = H$ (up to a constant).

If $\Phi = (T_k^1)^*\varphi$ for some $\varphi: Q \rightarrow Q$, then the Cartan symmetry Φ is said to be natural.

2. An infinitesimal Cartan (or Noether) symmetry is a vector field $Y \in \mathfrak{X}((T_k^1)^*Q)$ satisfying that:

- (a) $L(Y)\omega^A = 0$, for $A = 1, \dots, k$.
- (b) $L(Y)H = 0$.

If $Y = Z^C$ for some $Z \in \mathfrak{X}(Q)$, then the infinitesimal Cartan symmetry Y is said to be natural.

Furthermore, we have that:

Proposition 4 If $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$ is a Cartan symmetry of a k -symplectic Hamiltonian system $((T_k^1)^*Q, \omega^A, H)$, and $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$, then $\Phi_*\mathbf{X} = (\Phi_*X_1, \dots, \Phi_*X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$.

(Proof) Let $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$ be a Cartan symmetry. For every $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$ we calculate

$$\Phi^*[\sum_{A=1}^k i(\Phi_* X_A) \omega^A - dH] = \sum_{A=1}^k i(X_A)(\Phi^* \omega^A) - d(\Phi^* H) = \sum_{A=1}^k i(X_A) \omega^A - dH = 0$$

hence, as Φ is a diffeomorphism, this is equivalent to demanding that $\sum_{A=1}^k i(\Phi_* X_A) \omega^A - dH = 0$, and therefore $\Phi_* \mathbf{X} = (\Phi_* X_1, \dots, \Phi_* X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$. \blacksquare

In order to state a geometrical version of Noether's theorem for k -symplectic systems, we restrict our study to the infinitesimal Cartan symmetries.

First, it is immediate to prove that, if $Y_1, Y_2 \in \mathfrak{X}((T_k^1)^*Q)$ are infinitesimal Cartan symmetries, then so is $[Y_1, Y_2]$.

In addition, a highly relevant result is the following:

Proposition 5 *Let $Y \in \mathfrak{X}((T_k^1)^*Q)$ be an infinitesimal Cartan symmetry of a k -symplectic Hamiltonian system $((T_k^1)^*Q, \omega^A, H)$. Then, for $A = 1, \dots, k$, and for every $p \in (T_k^1)^*Q$, there is an open neighbourhood $U_p \ni p$, such that:*

1. *There exist $f^A \in C^\infty(U_p)$, which are unique up to constant functions, such that*

$$i(Y) \omega^A = df^A, \quad (\text{on } U_p). \quad (13)$$

2. *There exist $\zeta^A \in C^\infty(U_p)$, verifying that $L(Y) \theta^A = d\zeta^A$, on U_p ; and then*

$$f^A = i(Y) \theta^A - \zeta^A, \quad (\text{up to a constant function, on } U_p). \quad (14)$$

(Proof)

1. It is a consequence of the Poincaré Lemma and the condition

$$0 = L(Y) \omega^A = i(Y) d\omega^A + d i(Y) \omega^A = d i(Y) \omega^A.$$

2. We have that

$$d L(Y) \theta^A = L(Y) d\theta^A = -L(Y) \omega^A = 0$$

and hence $L(Y) \theta^A$ are closed forms. Therefore, by the Poincaré Lemma, there exist $\zeta^A \in C^\infty(U_p)$, verifying that $L(Y) \theta^A = d\zeta^A$, on U_p . Furthermore, as (13) holds on U_p , we obtain that

$$d\zeta^A = L(Y) \theta^A = d i(Y) \theta^A + i(Y) d\theta^A = d i(Y) \theta^A - i(Y) \omega^A = d\{i(Y) \theta^A - f^A\}$$

and thus (14) holds. \blacksquare

Remark: As a particular case, those Cartan symmetries $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$ (resp. infinitesimal Cartan symmetries $Y \in \mathfrak{X}((T_k^1)^*Q)$) verifying that $\Phi^* \theta^A = \theta^A$ (resp. $L(Y) \theta^A = 0$), for $A = 1, \dots, k$, are usually called *exact*. It is obvious that natural Cartan symmetries are exact.

Observe that, for exact infinitesimal Cartan symmetries we have that $f^A = -i(Y) \theta^A$.

Finally, the classical *Noether's theorem* of Hamiltonian mechanics can be generalized to k -symplectic field theories as follows:

Theorem 1 (Noether's theorem): *If $Y \in \mathfrak{X}((T_k^1)^*Q)$ is an infinitesimal Cartan symmetry of a k -symplectic Hamiltonian system $((T_k^1)^*Q, \omega^A, H)$. Then, for every $p \in (T_k^1)^*Q$, there is an open neighborhood $U_p \ni p$ such that the functions $f^A = i(Y)\theta^A - \zeta^A$, $1 \leq A \leq k$, define a conservation law $f = (f^1, \dots, f^k)$.*

(Proof) Let $Y \in \mathfrak{X}((T_k^1)^*Q)$ with local expression $Y = Y^i \frac{\partial}{\partial q^i} + Y_i^A \frac{\partial}{\partial p_i^A}$, then from (13) we have

$$Y^i \delta_B^A = \frac{\partial f^A}{\partial p_i^B} \quad , \quad -Y_i^A = \frac{\partial f^A}{\partial q^i} \quad ; \quad (\text{on } U_p)$$

Let $\psi: \mathbb{R}^k \rightarrow (T_k^1)^*Q$ be a solution to (4), then using the last equalities we obtain

$$\begin{aligned} \sum_{A=1}^k \frac{\partial(f^A \circ \psi)}{\partial t^A} \Big|_t &= \left(\frac{\partial f^A}{\partial q^i} \Big|_{\psi(t)} \frac{\partial \psi^i}{\partial t^A} \Big|_t + \frac{\partial f^A}{\partial p_i^B} \Big|_{\psi(t)} \frac{\partial \psi_i^B}{\partial t^A} \Big|_t \right) = \left(-Y_i^A \frac{\partial \psi^i}{\partial t^A} \Big|_t + Y^i \sum_{A=1}^k \frac{\partial \psi_i^A}{\partial t^A} \Big|_t \right) \\ &= - \left(Y_i^A \frac{\partial H}{\partial p_i^A} + Y^i \frac{\partial H}{\partial q^i} \right) = -L(Y)H = 0 \quad . \end{aligned}$$

■

In the case $k = 1$, the above theorem (Noether's Theorem in the Hamiltonian formalism) can be found in [34].

Furthermore, we have that:

Theorem 2 (Noether): *If $Y \in \mathfrak{X}((T_k^1)^*Q)$ is an infinitesimal Cartan symmetry of a k -symplectic Hamiltonian system $((T_k^1)^*Q, \omega^A, H)$. Then, for every $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$, we have*

$$\sum_{A=1}^k L(X_A) f^A = 0 \quad (\text{on } U_p) \quad .$$

(Proof) If $Y \in \mathfrak{X}((T_k^1)^*Q)$ is a Cartan-Noether symmetry, then, on U_p , taking (13) into account we obtain

$$\begin{aligned} \sum_{A=1}^k L(X_A) f^A &= \sum_{A=1}^k (d i(X_A) f^A + i(X_A) d f^A) = \sum_{A=1}^k i(X_A) i(Y) \omega^A \\ &= -i(Y) \sum_{A=1}^k i(X_A) \omega^A = -i(Y) dH = -L(Y)H = 0 \quad . \end{aligned}$$

■

Noether's theorem associates conservation laws to Cartan symmetries. However, these kinds of symmetries do not exhaust the set of symmetries. As is known, in mechanics there are symmetries which are not of Cartan type, and which also generate conserved quantities (see [31], [45], [46], for some examples). These are the so-called *hidden symmetries*. Different attempts have been made to extend Noether's theorem in order to include these symmetries and the corresponding conserved quantities for mechanical systems (see for instance [47]) and multisymplectic field theories (see [10]).

3 Lagrangian k -symplectic case

3.1 Geometric elements

3.1.1 The tangent bundle of k^1 -velocities of a manifold. Canonical structures

Let $\tau_Q: TQ \rightarrow Q$ be the tangent bundle of a Q . Let us denote by $T_k^1 Q$ the Whitney sum $TQ \oplus \dots \oplus TQ$ of k copies of TQ , with projection $\tau: T_k^1 Q \rightarrow Q$, $\tau(v_{1q}, \dots, v_{kq}) = q$.

$T_k^1 Q$ can be identified with the manifold $J_0^1(\mathbb{R}^k, Q)$ of the k^1 -velocities of Q ; that is, 1-jets of maps $\sigma: \mathbb{R}^k \rightarrow Q$, with source at $0 \in \mathbb{R}^k$ and with projection map $\tau: T_k^1 Q \rightarrow Q$, $\tau(j_{0,q}^1 \sigma) = \sigma(0) = q$; that is,

$$\begin{aligned} J_0^1(\mathbb{R}^k, Q) &\equiv TQ \oplus \dots \oplus TQ \\ j_{0,q}^1 \sigma &\equiv (v_{1q}, \dots, v_{kq}) \end{aligned}$$

where $q = \sigma(0)$, and $v_{Aq} = \sigma_*(0) \left(\frac{\partial}{\partial t^A} \Big|_0 \right)$. The manifold $T_k^1 Q$ is called *the tangent bundle of k^1 -velocities of Q* [38].

If (q^i) are local coordinates on $U \subseteq Q$ then the induced local coordinates (q^i, v^i) , $1 \leq i \leq n$, in $TU = \tau_Q^{-1}(U)$ are given by $q^i(v_q) = q^i(q)$, $v^i(v_q) = v_q(q^i)$, and the induced local coordinates (q^i, v_A^i) , $1 \leq i \leq n$, $1 \leq A \leq k$, in $T_k^1 U = \tau^{-1}(U)$ are given by

$$q^i(v_{1q}, \dots, v_{kq}) = q^i(q), \quad v_A^i(v_{1q}, \dots, v_{kq}) = v_{Aq}(q^i).$$

For a vector $Z_q \in T_q Q$, and for $A = 1, \dots, k$, we define its *vertical A -lift*, $(Z_q)^{VA}$, at the point $(v_{1q}, \dots, v_{kq}) \in T_k^1 Q$, as the vector tangent to the fiber $\tau^{-1}(q) \subset T_k^1 Q$, which is given by

$$(Z_q)^{VA}(v_{1q}, \dots, v_A) = \frac{d}{ds}(v_{1q}, \dots, v_{A-1q}, v_{Aq} + sZ_q, v_{A+1q}, \dots, v_{kq})|_{s=0}.$$

In local coordinates, if $X_q = a^i \frac{\partial}{\partial q^i} \Big|_q$, then

$$(Z_q)^{VA}(v_{1q}, \dots, v_{kq}) = a^i \frac{\partial}{\partial v_A^i} \Big|_{(v_{1q}, \dots, v_{kq})}. \quad (15)$$

The *canonical k -tangent structure* on $T_k^1 Q$ is the set (S^1, \dots, S^k) of tensor fields of type $(1, 1)$ defined by

$$S^A(w_q)(Z_{w_q}) = (\tau_*(w_q)(Z_{w_q}))^{VA}(w_q) \quad , \quad \text{for } w_q \in T_k^1 Q, Z_{w_q} \in T_{w_q}(T_k^1 Q); A = 1, \dots, k.$$

In local coordinates, from (15) we have

$$S^A = \frac{\partial}{\partial v_A^i} \otimes dq^i. \quad (16)$$

The tensors S^A can be regarded as the $(0, \dots, 0, \overset{A}{1}, 0, \dots, 0)$ -lift of the identity tensor on Q to $T_k^1 Q$ defined in [38]. In the case $k = 1$, S^1 is the well-known canonical tangent structure of the tangent bundle, (see [9, 19, 20, 26]).

Finally, we introduce the *Liouville vector field* $\Delta \in \mathfrak{X}(T_k^1 Q)$, which is the infinitesimal generator of the following flow

$$\psi: \mathbb{R} \times T_k^1 Q \longrightarrow T_k^1 Q \quad , \quad \psi(s, v_{1q}, \dots, v_{kq}) = (e^s v_{1q}, \dots, e^s v_{kq}),$$

and in local coordinates it has the form

$$\Delta = \sum_{A=1}^k v_A^i \frac{\partial}{\partial v_A^i} .$$

Δ is a sum of vector fields $\Delta_1 + \dots + \Delta_k$, where each Δ_A is the infinitesimal generator of the following flow

$$\psi^A: \mathbb{R} \times T_k^1 Q \longrightarrow T_k^1 Q \quad , \quad \psi^A(s, v_{1q}, \dots, v_{kq}) = (v_{1q}, \dots, v_{A-1q}, e^s v_{Aq}, v_{A+1q}, \dots, v_{kq}) \quad (17)$$

and, in local coordinates, each Δ_A has the form

$$\Delta_A = v_A^i \frac{\partial}{\partial v_A^i} \quad , \quad \text{for } A = 1, \dots, k \text{ fixed.} \quad (18)$$

3.1.2 Complete lift of diffeomorphisms and vector fields from Q to $T_k^1 Q$

Let $\varphi: Q \rightarrow Q$ be a differentiable map, then the *canonical prolongation* of φ to $T_k^1 Q$ is the induced map $T_k^1 \varphi: T_k^1 Q \rightarrow T_k^1 Q$ defined by $T_k^1 \varphi(j_0^1 \sigma) = j_0^1(\varphi \circ \sigma)$; that is, for $v_{1q}, \dots, v_{kq} \in T_q Q$, $q \in Q$.

$$T_k^1 \varphi(v_{1q}, \dots, v_{kq}) = (\varphi_*(q)v_{1q}, \dots, \varphi_*(q)v_{kq}) .$$

If Z is a vector field on Q , with local 1-parametric group of transformations $h_s: Q \rightarrow Q$, then the local 1-parametric group of transformations $T_k^1(h_s): T_k^1 Q \rightarrow T_k^1 Q$ generates a vector field Z^C on $T_k^1 Q$, which is called the *complete lift* of Z to $T_k^1 Q$. If where $Z = Z^i \frac{\partial}{\partial q^i}$, its local expression is

$$Z^C = Z^i \frac{\partial}{\partial q^i} + v_A^j \frac{\partial Z^k}{\partial q^j} \frac{\partial}{\partial v_A^k} .$$

Then, we have the following property:

Lemma 2 *Let $\Phi = T_k^1 \varphi: T_k^1 Q \rightarrow T_k^1 Q$ be the canonical prolongation of a diffeomorphism $\varphi: Q \rightarrow Q$. Then*

$$(a) \quad \Phi_* \circ S^A = S^A \circ \Phi_* \quad , \quad (b) \quad \Phi_* \Delta_A = \Delta_A \quad , \quad \text{for } A = 1, \dots, k \quad .$$

(Proof) (a) It is a direct consequence of local expression of S^A and the local expression of $T_k^1 \varphi$,

$$T_k^1 \varphi(q^i, v_A^i) = (\varphi^j(q^i), v_A^i \frac{\partial \varphi^j}{\partial q^i}) .$$

(b) It is a consequence of $T_k^1 \varphi \circ \psi_t^A = \psi_t^A \circ T_k^1 \varphi$, where ψ_t^A are the local 1-parameter groups of diffeomorphisms (17) generated by Δ_A . ■

This means that canonical liftings of diffeomorphisms and vector fields preserve the canonical structures of $T_k^1 Q$.

3.1.3 Second-order partial differential equations in $T_k^1 Q$

The aim of this subsection is to characterize the integrable k -vector fields on $T_k^1 Q$ such that their integral sections are first prolongations $\phi^{(1)}$ of maps $\phi: \mathbb{R}^k \rightarrow Q$.

Remember that a k -vector field in $T_k^1 Q$ is a section $\Gamma: T_k^1 Q \rightarrow T_k^1(T_k^1 Q)$ of the canonical projection $\tau_{T_k^1 Q}: T_k^1(T_k^1 Q) \rightarrow T_k^1 Q$. Then:

Definition 6 A second order partial differential equation (SOPDE) is a k -vector field $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ in $T_k^1 Q$ which is a section of the projection $T_k^1 \tau: T_k^1(T_k^1 Q) \rightarrow T_k^1 Q$; that is,

$$T_k^1 \tau \circ \Gamma = \text{Id}_{T_k^1 Q} ,$$

or, what is equivalent,

$$\tau_*(w_q)(\Gamma_A(w_q)) = v_{Aq} \quad , \quad \text{for } w_q = (v_{1q}, \dots, v_{kq}) \in T_k^1 Q, A = 1, \dots, k .$$

In the case $k = 1$, this is the definition of a second order differential equation (SODE).

From a direct computation in local coordinates we obtain that the local expression of a SOPDE $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ is

$$\Gamma_A(q^i, v_A^i) = v_A^i \frac{\partial}{\partial q^i} + (\Gamma_A)_B^i \frac{\partial}{\partial v_B^i} \quad , \quad 1 \leq A \leq k \quad , \quad (\Gamma_A)_B^i \in C^\infty(T_k^1 Q) . \quad (19)$$

If $\psi: \mathbb{R}^k \rightarrow T_k^1 Q$ is an integral section of $\Gamma = (\Gamma_1, \dots, \Gamma_k)$, locally given by $\psi(t) = (\psi^i(t), \psi_B^i(t))$, then from Definition 2 and (19) we deduce

$$\left. \frac{\partial \psi^i}{\partial t^A} \right|_t = \psi_A^i(t) \quad , \quad \left. \frac{\partial \psi_B^i}{\partial t^A} \right|_t = (\Gamma_A)_B^i(\psi(t)) . \quad (20)$$

From (3) and (20) we obtain the following proposition.

Proposition 6 Let $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ be an integrable SOPDE. If ψ is an integral section of Γ then $\psi = \phi^{(1)}$, where $\phi^{(1)}$ is the first prolongation of the map $\phi = \tau \circ \psi: \mathbb{R}^k \xrightarrow{\psi} T_k^1 Q \xrightarrow{\tau} Q$, and ϕ is a solution to the system of second order partial differential equations

$$\frac{\partial^2 \phi^i}{\partial t^A \partial t^B}(t) = (\Gamma_A)_B^i \left(\phi^i(t), \frac{\partial \phi^i}{\partial t^C}(t) \right) \quad 1 \leq i \leq n ; 1 \leq A, B \leq k . \quad (21)$$

Conversely, if $\phi: \mathbb{R}^k \rightarrow Q$ is any map satisfying (21), then $\phi^{(1)}$ is an integral section of $\Gamma = (\Gamma_1, \dots, \Gamma_k)$.

From (21) we deduce that if Γ is an integrable SOPDE then $(\Gamma_A)_B^i = (\Gamma_B)_A^i$ for all $A, B = 1, \dots, k$.

The following characterization of SOPDEs can be given using the canonical k -tangent structure of $T_k^1 Q$ (see (16), (18) and (19)):

Proposition 7 A k -vector field $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ on $T_k^1 Q$ is a SOPDE if, and only if, $S^A(\Gamma_A) = \Delta_A$, for all $A: 1 \dots, k$.

3.2 Lagrangian formalism: k -symplectic Lagrangian systems

In Classical Mechanics, the symplectic structure of Hamiltonian theory and the tangent structure of Lagrangian theory play complementary roles (see Refs. [13,15,16]). In this subsection, we recall the Lagrangian formalism developed by Günther [21] using the polysymplectic structures. Here we can see how the polysymplectic structures and the k -tangent structures also play a complementary role in field theory.

Let $L: T_k^1 Q \rightarrow \mathbb{R}$ be a Lagrangian. The *generalized Euler-Lagrange equations* for L are:

$$\sum_{A=1}^k \frac{\partial}{\partial t^A} \Big|_t \left(\frac{\partial L}{\partial v_A^i} \Big|_{\psi(t)} \right) = \frac{\partial L}{\partial q^i} \Big|_{\psi(t)} \quad , \quad v_A^i(\psi(t)) = \frac{\partial \psi^i}{\partial t^A} \quad (22)$$

whose solutions are maps $\psi: \mathbb{R}^k \rightarrow T_k^1 Q$. Let us observe that $\psi(t) = \phi^{(1)}(t)$, for some $\phi = \tau \circ \psi$.

We introduce a family of 1-forms θ_L^A on $T_k^1 Q$, $1 \leq A \leq k$, using the k -tangent structure, as follows

$$\theta_L^A = dL \circ S^A \quad 1 \leq A \leq k \quad , \quad (23)$$

and hence we define $\omega_L^A = -d\theta_L^A$.

In local natural coordinates we have

$$\theta_L^A = \frac{\partial L}{\partial v_A^i} dq^i \quad (24)$$

$$\omega_L^A = dq^i \wedge d \left(\frac{\partial L}{\partial v_A^i} \right) = \frac{\partial^2 L}{\partial q^j \partial v_A^i} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} dq^i \wedge dv_B^j . \quad (25)$$

We also introduce the *Energy lagrangian function* $E_L = \Delta(L) - L \in C^\infty(T_k^1 Q)$, whose local expression is

$$E_L = v_A^i \frac{\partial L}{\partial v_A^i} - L . \quad (26)$$

Then, the family $(T_k^1 Q, \omega_L^A, E_L)$ is called a *k -symplectic Lagrangian system*.

Definition 7 *The Lagrangian $L: T_k^1 Q \rightarrow \mathbb{R}$ is said to be regular if the matrix $\left(\frac{\partial^2 L}{\partial v_A^i \partial v_B^j} \right)$ is not singular at every point of $T_k^1 Q$.*

Remark: Let us observe that the condition L regular is equivalent to $(\omega_L^1, \dots, \omega_L^k)$ being a polysymplectic form and $(\omega_L^1, \dots, \omega_L^k; V)$, where $V = \text{Ker} \tau_*$, is a k -symplectic structure (see [37]).

This k -symplectic (polysymplectic) structure, associated to L , was also introduced by Günther [21] using the Legendre transformation.

The *Legendre map* $FL: T_k^1 Q \rightarrow (T_k^1 Q)^* Q$ was introduced by Günther, [21] and was rewritten in [37] as follows: if $(v_{1_q}, \dots, v_{k_q}) \in (T_k^1)_q Q$,

$$[FL(v_{1_q}, \dots, v_{k_q})]^A(u_q) = \frac{d}{ds} \Big|_{s=0} L(v_{1_q}, \dots, v_{A_q} + s u_q, \dots, v_{k_q}) ,$$

for each $A = 1, \dots, k$ and $u_q \in T_q Q$. Locally FL is given by

$$FL(q^i, v_A^i) = (q^i, \frac{\partial L}{\partial v_A^i}) . \quad (27)$$

In fact, from (24) and (27), we easily obtain the following Lemma.

Lemma 3 *For every $1 \leq A \leq k$, $\omega_L^A = (FL)^* \omega^A$, where $(\omega^1, \dots, \omega^k)$ are the 2-forms of the canonical polysymplectic structure.*

Then, from (27) we obtain the following Proposition.

Proposition 8 *Let L be a Lagrangian. The following conditions are equivalent: (1) L is regular. (2) FL is a local diffeomorphism. (3) $(\omega_L^1, \dots, \omega_L^k)$ is a polysymplectic structure on $T_k^1 Q$.*

As in the Hamiltonian case, consider a k -symplectic Lagrangian system $(T_k^1 Q, \omega_L^A, E_L)$, and denote by $\mathfrak{X}_L^k(T_k^1 Q)$ the set of k -vector fields $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ in $T_k^1 Q$, which are solutions to the equation

$$\sum_{A=1}^k i(\Gamma_A) \omega_L^A = dE_L . \quad (28)$$

If each Γ_A is locally given by

$$\Gamma_A = (\Gamma_A)^i \frac{\partial}{\partial q^i} + (\Gamma_A)^i_B \frac{\partial}{\partial v_B^i} ,$$

then $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ is a solution to (28) if, and only if, $(\Gamma_A)^i$ and $(\Gamma_A)^i_B$ satisfy the system of equations

$$\begin{aligned} \left(\frac{\partial^2 L}{\partial q^i \partial v_A^j} - \frac{\partial^2 L}{\partial q^j \partial v_A^i} \right) (\Gamma_A)^j - \frac{\partial^2 L}{\partial v_A^i \partial v_B^j} (\Gamma_A)^j_B &= v_A^j \frac{\partial^2 L}{\partial q^i \partial v_A^j} - \frac{\partial L}{\partial q^i} , \\ \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} (\Gamma_A)^i &= \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} v_A^i . \end{aligned}$$

If the Lagrangian is regular, the above equations are equivalent to the equations

$$\frac{\partial^2 L}{\partial q^j \partial v_A^i} v_A^j + \frac{\partial^2 L}{\partial v_A^i \partial v_B^j} (\Gamma_A)^j_B = \frac{\partial L}{\partial q^i} \quad (29)$$

$$(\Gamma_A)^i = v_A^i , \quad 1 \leq i \leq n, 1 \leq A \leq k . \quad (30)$$

Thus, if L is a regular Lagrangian, we deduce:

- If $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ is a solution to (28) then it is a SOPDE, (see (30)).
- Equation (29) leads to define local solutions to (28) in a neighborhood of each point of $T_k^1 Q$ and, using a partition of unity, global solutions to (28).
- Since $\Gamma = (\Gamma_1, \dots, \Gamma_k) \in \mathfrak{X}_L^k(T_k^1 Q)$ is a SOPDE, from Proposition 6 we know that, if it is integrable, then its integral sections are first prolongations $\phi^{(1)}: \mathbb{R}^k \rightarrow T_k^1 Q$ of maps $\phi: \mathbb{R}^k \rightarrow Q$, and from (29) we deduce that ϕ is a solution to the Euler-Lagrange equations (22).

- In the case $k = 1$, the equation (28) is $\iota_{\Gamma}\omega_L = dE_L$, which is the dynamical equation of the Lagrangian formalism in Mechanics.

Throughout this paper, we only consider regular Lagrangians.

3.3 Symmetries and conservation laws

Of course, regarding these topics, of course, all the definitions stated in Section 2.3 for the Hamiltonian case are applied to the Lagrangian case, just considering $(T_k^1Q, \omega_L^A, E_L)$ as a Hamiltonian system with Hamiltonian function E_L . In particular, we can define:

Definition 8 *A map $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k): T_k^1Q \rightarrow \mathbb{R}^k$ is a conservation law (or a conserved quantity) for the Euler-Lagrange equations (22) if the divergence of $\mathcal{F} \circ \phi = (\mathcal{F}^1 \circ \phi^{(1)}, \dots, \mathcal{F}^k \circ \phi^{(1)}): \mathbb{R}^k \rightarrow \mathbb{R}^k$ is zero, for every $\phi: \mathbb{R}^k \rightarrow Q$ solution to the Euler-Lagrange equations (22); that is*

$$\sum_{A=1}^k \frac{\partial(\mathcal{F}^A \circ \phi^{(1)})}{\partial t^A} = 0.$$

Therefore, if $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k): T_k^1Q \rightarrow \mathbb{R}^k$ is a conservation law then, for every integrable k -vector field $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ in $\mathfrak{X}_L^k(T_k^1Q)$, we have that

$$\sum_{A=1}^k L(\Gamma_A) \mathcal{F}^A = 0.$$

Definition 9 *1. A symmetry of the k -symplectic Lagrangian system $(T_k^1Q, \omega_L^A, E_L)$ is a diffeomorphism $\Phi: T_k^1Q \rightarrow T_k^1Q$ such that, for every solution ϕ to the Euler-Lagrange equations (22), we have that $\Phi \circ \phi^{(1)} = \rho^{(1)}$, where $\rho: \mathbb{R}^k \rightarrow Q$ is also a solution to these equations.*

In the particular case that $\Phi = T_k^1\varphi$ for some $\varphi: Q \rightarrow Q$ (i.e.; Φ is the canonical lifting of some diffeomorphism in Q), the symmetry Φ is said to be natural.

2. An infinitesimal symmetry of the k -symplectic Lagrangian system $(T_k^1Q, \omega_L^A, E_L)$ is a vector field $Y \in \mathfrak{X}(T_k^1Q)$ whose local flows are local symmetries.

In the particular case that $Y = Z^C$ for some $Z \in \mathfrak{X}(Q)$, (i.e.; Y is the canonical lifting of some vector field in Q), the infinitesimal symmetry Y is said to be natural.

As in the Hamiltonian case, we have that:

Proposition 9 *Let $\Phi: T_k^1Q \rightarrow T_k^1Q$ be a diffeomorphism. If Φ satisfies*

$$\Phi^* \omega_L^A = \omega_L^A, \quad 1 \leq A \leq k \quad \text{and} \quad \Phi^* E_L = E_L \quad (\text{up to a constant}).$$

then Φ is a symmetry of the k -symplectic Lagrangian system $(T_k^1Q, \omega_L^A, E_L)$.

(Proof) We must prove that, if $\phi: U_0 \subset \mathbb{R}^k \rightarrow Q$ is a solution to the Euler-Lagrange equations (22), then $\Phi \circ \phi^{(1)}$ is also a solution. However, it is well-known that this is equivalent to proving

that $FL \circ \Phi \circ \phi^{(1)}: U_0 \subset \mathbb{R}^k \rightarrow (T_k^1)^*Q$ is a solution to the Hamilton-de Donder-Weyl equations, (4); that is

$$\begin{aligned} (a) \quad \frac{\partial H}{\partial p_i^A} \Big|_{(FL \circ \Phi \circ \phi^{(1)})(t)} &= \frac{\partial (FL \circ \Phi \circ \phi^{(1)})^i}{\partial t^A} \Big|_t \\ (b) \quad \frac{\partial H}{\partial q^i} \Big|_{(FL \circ \Phi \circ \phi^{(1)})(t)} &= - \sum_{A=1}^k \frac{\partial (FL \circ \Phi \circ \phi^{(1)})_A^i}{\partial t^A} \Big|_t, \end{aligned}$$

with Hamiltonian $H = E_L \circ FL^{-1}$.

Let us suppose that $\Phi: T_k^1 Q \rightarrow T_k^1 Q$, locally given by $\Phi(q^j, v_B^j) = (\Phi^i(q^j, v_B^j), \Phi_A^i(q^j, v_B^j))$ satisfies the conditions $\Phi^* \omega_L^A = \omega_L^A$ and $E_L = \Phi^* E_L$.

In order to prove (a) and (b) we will use four groups of identities. From the condition $\Phi^* \omega_L^A = \omega_L^A$ we obtain the first group of identities: for every $w \in T_k^1 Q$,

$$\begin{aligned} \frac{\partial^2 L}{\partial q^j \partial v_A^i} \Big|_w &= \left(\frac{\partial^2 L}{\partial q^k \partial v_A^l} \Big|_{\Phi(w)} \frac{\partial \Phi^k}{\partial q^j} \Big|_w + \frac{\partial^2 L}{\partial v_C^k \partial v_A^l} \Big|_{\Phi(w)} \frac{\partial \Phi_C^k}{\partial q^j} \Big|_w \right) \frac{\partial \Phi^l}{\partial q^i} \Big|_w, \\ \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} \Big|_w &= \left(\frac{\partial^2 L}{\partial q^k \partial v_A^l} \Big|_{\Phi(w)} \frac{\partial \Phi^k}{\partial v_B^j} \Big|_w + \frac{\partial^2 L}{\partial v_C^k \partial v_A^l} \Big|_{\Phi(w)} \frac{\partial \Phi_C^k}{\partial v_B^j} \Big|_w \right) \frac{\partial \Phi^l}{\partial q^i} \Big|_w \\ &\quad - \left(\frac{\partial^2 L}{\partial q^k \partial v_A^l} \Big|_{\Phi(w)} \frac{\partial \Phi^k}{\partial q^i} \Big|_w + \frac{\partial^2 L}{\partial v_C^k \partial v_A^l} \Big|_{\Phi(w)} \frac{\partial \Phi_C^k}{\partial q^i} \Big|_w \right) \frac{\partial \Phi^l}{\partial v_B^j} \Big|_w, \\ 0 &= \left(\frac{\partial^2 L}{\partial q^k \partial v_A^l} \Big|_{\Phi(w)} \frac{\partial \Phi^k}{\partial v_B^j} \Big|_w + \frac{\partial^2 L}{\partial v_C^k \partial v_A^l} \Big|_{\Phi(w)} \frac{\partial \Phi_C^k}{\partial v_B^j} \Big|_w \right) \frac{\partial \Phi^l}{\partial v_D^m} \Big|_w. \end{aligned} \quad (31)$$

Applying the chain rule to $\Phi \circ \Phi^{-1} = Id_{T_k^1 Q}$, we have the second group.

$$\delta_k^i = \frac{\partial \Phi^i}{\partial q^j} \Big|_{\Phi^{-1}(w)} \frac{\partial (\Phi^{-1})^j}{\partial q^k} \Big|_w + \frac{\partial \Phi^i}{\partial v_A^j} \Big|_{\Phi^{-1}(w)} \frac{\partial (\Phi^{-1})_A^j}{\partial q^k} \Big|_w, \quad (32)$$

$$0 = \frac{\partial \Phi^i}{\partial q^j} \Big|_{\Phi^{-1}(w)} \frac{\partial (\Phi^{-1})^j}{\partial v_B^k} \Big|_w + \frac{\partial \Phi^i}{\partial v_A^j} \Big|_{\Phi^{-1}(w)} \frac{\partial (\Phi^{-1})_A^j}{\partial v_B^k} \Big|_w, \quad (33)$$

$$0 = \frac{\partial \Phi_A^i}{\partial q^k} \Big|_{\Phi^{-1}(w)} \frac{\partial (\Phi^{-1})^k}{\partial q^j} \Big|_w + \frac{\partial \Phi_A^i}{\partial v_B^k} \Big|_{\Phi^{-1}(w)} \frac{\partial (\Phi^{-1})_B^k}{\partial q^j} \Big|_w. \quad (34)$$

The third group of identities is a consequence of the following fact: if $\phi: U_0 \subset \mathbb{R}^k \rightarrow Q$ is a solution to Euler-Lagrange's equations, we know that $FL \circ \phi^{(1)}: U_0 \subset \mathbb{R}^k \rightarrow (T_k^1)^*Q$ is a solution to Hamilton-de Donder-Weyl's equations (4). Then from the local expression of FL , (27) we deduce the following equations.

$$\begin{aligned} \frac{\partial H}{\partial p_i^A} \Big|_{(FL \circ \phi^{(1)})(t)} &= \frac{\partial (FL \circ \phi^{(1)})^i}{\partial t^A} \Big|_t = \frac{\partial \phi^i}{\partial t^A} \Big|_t, \\ \frac{\partial H}{\partial q^i} \Big|_{(FL \circ \phi^{(1)})(t)} &= - \sum_{A=1}^k \frac{\partial (FL \circ \phi^{(1)})_A^i}{\partial t^A} \Big|_t \\ &= - \frac{\partial^2 L}{\partial q^j \partial v_A^i} \Big|_{\phi^{(1)}(t)} \frac{\partial \phi^j}{\partial t^A} \Big|_t - \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} \Big|_{\phi^{(1)}(t)} \frac{\partial^2 \phi^j}{\partial t^A \partial t^B} \Big|_t. \end{aligned} \quad (35)$$

Since $E_L = \Phi^* E_L$ is equivalent to $FL^* H = (FL \circ \Phi)^* H$, by applying the chain rule again and by using the local expression of FL (27), we obtain the last family of identities

$$\begin{aligned} \frac{\partial H}{\partial q^i} \Big|_{FL(w)} + \frac{\partial H}{\partial p_j^B} \Big|_{FL(w)} \frac{\partial^2 L}{\partial q^i \partial v_B^j} \Big|_w &= \frac{\partial H}{\partial q^j} \Big|_{(FL \circ \Phi)(w)} \frac{\partial \Phi^j}{\partial q^i} \Big|_w \\ &+ \frac{\partial H}{\partial p_j^B} \Big|_{(FL \circ \Phi)(w)} \left(\frac{\partial^2 L}{\partial q^k \partial v_B^j} \Big|_{\Phi(w)} \frac{\partial \Phi^k}{\partial q^i} \Big|_w + \frac{\partial^2 L}{\partial v_A^k \partial v_B^j} \Big|_{\Phi(w)} \frac{\partial \Phi_A^k}{\partial q^i} \Big|_w \right) \end{aligned} \quad (36)$$

$$\begin{aligned} \frac{\partial H}{\partial p_j^B} \Big|_{FL(w)} \frac{\partial^2 L}{\partial v_A^i \partial v_B^j} \Big|_w &= \frac{\partial H}{\partial q^j} \Big|_{(FL \circ \Phi)(w)} \frac{\partial \Phi^j}{\partial v_A^i} \Big|_w \\ &+ \frac{\partial H}{\partial p_j^B} \Big|_{(FL \circ \Phi)(w)} \left(\frac{\partial^2 L}{\partial q^k \partial v_B^j} \Big|_{\Phi(w)} \frac{\partial \Phi^k}{\partial v_A^i} \Big|_w + \frac{\partial^2 L}{\partial v_C^k \partial v_B^j} \Big|_{\Phi(w)} \frac{\partial \Phi_C^k}{\partial v_A^i} \Big|_w \right). \end{aligned} \quad (37)$$

These identities (36) and (37) are fundamental to proof of this proposition. Let us observe that in these identities we find the partial derivatives $\frac{\partial H}{\partial q^i} \Big|_{(FL \circ \Phi \circ \phi^{(1)})(t)}$ and $\frac{\partial H}{\partial p_i^A} \Big|_{(FL \circ \Phi \circ \phi^{(1)})(t)}$, which we are searching for, and their relation with the other partial derivatives $\frac{\partial H}{\partial q^i} \Big|_{(FL \circ \phi^{(1)})(t)}$ and $\frac{\partial H}{\partial p_i^A} \Big|_{(FL \circ \phi^{(1)})(t)}$, which we know from (35).

By a straightforward computation, from equations (31-33), (35-37) one proves that

$$0 = \frac{\partial^2 L}{\partial v_D^s \partial v_A^l} \Big|_{(\Phi \circ \phi^{(1)})(t)} \left(\frac{\partial H}{\partial p_l^A} \Big|_{(FL \circ \Phi \circ \phi^{(1)})(t)} - \frac{\partial \Phi^l}{\partial q^j} \Big|_{\phi^{(1)}(t)} \frac{\partial \phi^j}{\partial t^A} \Big|_t - \frac{\partial \Phi^l}{\partial v_B^j} \Big|_{\phi^{(1)}(t)} \frac{\partial^2 \phi^j}{\partial t^A \partial t^B} \Big|_t \right)$$

and since L is regular, from the above identity we deduce that

$$\frac{\partial H}{\partial p_l^A} \Big|_{(FL \circ \Phi \circ \phi^{(1)})(t)} = \frac{\partial \Phi^l}{\partial q^j} \Big|_{\phi^{(1)}(t)} \frac{\partial \phi^j}{\partial t^A} \Big|_t + \frac{\partial \Phi^l}{\partial v_B^j} \Big|_{\phi^{(1)}(t)} \frac{\partial^2 \phi^j}{\partial t^A \partial t^B} \Big|_t. \quad (38)$$

Furthermore we have

$$\frac{\partial (FL \circ \Phi \circ \phi^{(1)})^l}{\partial t^A} \Big|_t = \frac{\partial \Phi^l}{\partial q^j} \Big|_{\phi^{(1)}(t)} \frac{\partial \phi^j}{\partial t^A} \Big|_t + \frac{\partial \Phi^l}{\partial v_B^j} \Big|_{\phi^{(1)}(t)} \frac{\partial^2 \phi^j}{\partial t^A \partial t^B} \Big|_t \quad (39)$$

and thus from (38) and (39) we obtain the first group, (a), of the Hamilton-de Donder-Weyl equations.

Finally, from (31), (32), (34-37) and (39), by a straightforward computation, one obtains

$$\begin{aligned} \sum_{A=1}^k \frac{\partial (FL \circ \Phi \circ \phi^{(1)})_A^m}{\partial t^A} \Big|_t &= - \frac{\partial H}{\partial q^m} \Big|_{(FL \circ \Phi \circ \phi^{(1)})(t)} \\ &+ \frac{\partial^2 L}{\partial q^m \partial v_A^i} \Big|_{(\Phi \circ \phi^{(1)})(t)} \left(\frac{\partial (FL \circ \Phi \circ \phi^{(1)})^i}{\partial t^A} \Big|_t - \frac{\partial H}{\partial p_i^A} \Big|_{(FL \circ \Phi \circ \phi^{(1)})(t)} \right) \end{aligned} \quad (40)$$

and since we have already proved (a), from (40) and (a) one obtains (b). \blacksquare

Taking into account this proposition, we introduce the following definitions.

Definition 10 1. A Cartan (or Noether) symmetry of the k -symplectic Lagrangian system $(T_k^1 Q, \omega_L^A, E_L)$ is a diffeomorphism $\Phi: T_k^1 Q \rightarrow T_k^1 Q$ such that,

- (a) $\Phi^* \omega_L^A = \omega_L^A$, for $A = 1, \dots, k$.
- (b) $\Phi^* E_L = E_L$ (up to a constant).

If $\Phi = T_k^1 \varphi$ for some $\varphi: Q \rightarrow Q$, then the Cartan symmetry Φ is said to be natural.

2. An infinitesimal Cartan (or Noether) symmetry of the k -symplectic Lagrangian system $(T_k^1 Q, \omega_L^A, E_L)$ is a vector field $Y \in \mathfrak{X}(T_k^1 Q)$ satisfying that:

- (a) $L(Y) \omega_L^A = 0$, for $A = 1, \dots, k$.
- (b) $L(Y) E_L = 0$.

If $Y = Z^C$ for some $Z \in \mathfrak{X}(Q)$, then the infinitesimal Cartan symmetry Y is said to be natural.

Proposition 10 *Let $Y \in \mathfrak{X}(T_k^1 Q)$ be an infinitesimal Cartan symmetry of a k -symplectic Lagrangian system $(T_k^1 Q, \omega_L^A, E_L)$. Then, for $A = 1, \dots, k$, and for every $p \in (T_k^1 Q)$, there is an open neighborhood $U_p \ni p$, such that:*

1. There exist $f^A \in C^\infty(U_p)$, which are unique up to constant functions, such that

$$i(Y) \omega_L^A = df^A, \quad (\text{on } U_p). \quad (41)$$

2. There exist $\zeta^A \in C^\infty(U_p)$, verifying that $L(Y) \theta_L^A = d\zeta^A$, on U_p ; and then

$$f^A = i(Y) \theta_L^A - \zeta^A, \quad (\text{up to a constant function, on } U_p). \quad (42)$$

(Proof) This is the same proof as in Proposition 5. ■

Now we can state the version of Noether's Theorem for infinitesimal Cartan Lagrangian symmetries.

Theorem 3 (Noether's theorem): *Let $Y \in \mathfrak{X}(T_k^1 Q)$ be an infinitesimal Cartan symmetry of a k -symplectic Lagrangian system $(T_k^1 Q, \omega_L^A, E_L)$, then for every $p \in T_k^1 Q$, there is an open neighborhood $U_p \ni p$, such that the functions $f^A = i(Y) \theta_L^A - \zeta^A$, $1 \leq A \leq k$, define a conservation law $f = (f^1, \dots, f^k)$.*

(Proof) Let $Y \in \mathfrak{X}((T_k^1)^* Q)$ be an infinitesimal Cartan symmetry, with local expression

$$Y = Y^i \frac{\partial}{\partial q^i} + Y_A^i \frac{\partial}{\partial v_A^i}.$$

Then from (41), as Y is an infinitesimal Cartan symmetry we have that

$$\left(\frac{\partial^2 L}{\partial q^k \partial v_A^i} - \frac{\partial^2 L}{\partial q^i \partial v_A^k} \right) Y^i - Y_B^i \frac{\partial^2 L}{\partial v_B^i \partial v_A^k} = \frac{\partial f^A}{\partial q^k} \quad (43)$$

$$\frac{\partial^2 L}{\partial v_B^k \partial v_A^i} Y^i = \frac{\partial f^A}{\partial v_B^k}. \quad (44)$$

Therefore, since Y is an infinitesimal symmetry, from $L(Y) E_L = 0$ we obtain

$$Y^i \frac{\partial L}{\partial q^i} = v_B^k \left(Y^i \frac{\partial^2 L}{\partial q^i \partial v_B^k} + Y_A^i \frac{\partial^2 L}{\partial v_A^i \partial v_B^k} \right). \quad (45)$$

Let $\phi : \mathbb{R}^k \rightarrow Q$ be a solution to the Euler-Lagrange equations, then from (22), (43), (44) and (45) we obtain

$$\begin{aligned}
\sum_{A=1}^k \frac{\partial(f^A \circ \phi^{(1)})}{\partial t^A} \Big|_t &= \sum_{A=1}^k \left(\frac{\partial f^A}{\partial q^k} \Big|_{\phi^{(1)}(t)} \frac{\partial \phi^k}{\partial t^A} \Big|_t + \frac{\partial f^A}{\partial v_B^k} \Big|_{\phi^{(1)}(t)} \frac{\partial^2 \phi^k}{\partial t^A \partial t^B} \Big|_t \right) \\
&= \sum_{A=1}^k Y^i(\phi^{(1)}(t)) \left(\frac{\partial^2 L}{\partial q^k \partial v_A^i} \Big|_{\phi^{(1)}(t)} \frac{\partial \phi^k}{\partial t^A} \Big|_t + \frac{\partial^2 L}{\partial v_B^k \partial v_A^i} \Big|_{\phi^{(1)}(t)} \frac{\partial^2 \phi^k}{\partial t^A \partial t^B} \Big|_t \right) \\
&\quad - \sum_{A=1}^k \left(Y^i(\phi^{(1)}(t)) \frac{\partial^2 L}{\partial q^i \partial v_A^k} \Big|_{\phi^{(1)}(t)} + Y_B^i(\phi^{(1)}(t)) \frac{\partial^2 L}{\partial v_B^i \partial v_A^k} \Big|_{\phi^{(1)}(t)} \right) \frac{\partial \phi^k}{\partial t^A} \Big|_t \\
&= Y^i(\phi^{(1)}(t)) \frac{\partial L}{\partial q^i} \Big|_{\phi^{(1)}(t)} - Y^i(\phi^{(1)}(t)) \frac{\partial L}{\partial q^i} \Big|_{\phi^{(1)}(t)} = 0, \quad (\text{on } U_p).
\end{aligned}$$

■

Corollary 1 *If $Z^C \in \mathfrak{X}(T_k^1 Q)$ is an infinitesimal natural Cartan symmetry of a k-symplectic Lagrangian system $(T_k^1 Q, \omega_L^A, E_L)$ then the functions $f^A = Z^{V_A}(L) - \zeta^A$, $1 \leq A \leq k$, define a conservation law on U_p .*

(Proof) In this case, we have

$$i(Z^C)\theta_L^A = \theta_L^A(Z^C) = dL \circ S^A(Z^C) = dL(Z^{V_A}) = Z^{V_A}(L),$$

and thus the functions f^A of Proposition 10 can be written

$$f^A = Z^{V_A}(L) - \zeta^A, \quad 1 \leq A \leq k.$$

■

The case $k = 1$ corresponds to Classical Mechanics, and the above results can be found in [9].

Remark: The above Noether's theorem can be rewritten introducing the following generalization of the so-called *Tulczyjew operator* [49] for our case: Let $g = (g^1, \dots, g^k) : Q \rightarrow \mathbb{R}^k$ be a function, we define $d_T g : T_k^1 Q \rightarrow \mathbb{R}$ by

$$d_T g(v_{1q}, \dots, v_{kq}) = \sum_{A=1}^k v_{Aq}(g^A) = v_A^i \frac{\partial g^A}{\partial q^i}.$$

Then it is not difficult to prove that the condition $Z^C(L) = d_T g$ is equivalent to the conditions $L(Z^C)\theta_L^A = d\tau^* g^A$ and $Z^C(E_L) = 0$. Therefore, by comparing with item 2 in Proposition 10 we observe that the functions f^A can be written as

$$f^A = Z^{V_A}(L) - \tau^* g^A, \quad A = 1, \dots, k.$$

Therefore, we have the following proposition,

Proposition 11 *If $Z \in \mathfrak{X}(Q)$ and $Z^C(L) = d_T g$, where $g = (g^1, \dots, g^k) : Q \rightarrow \mathbb{R}^k$, then the functions $f^A = Z^{V_A}(L) - \tau^* g^A$, $1 \leq A \leq k$, define a conservation law.*

(*Proof*) This result is a consequence of Theorem 3. In fact, $Z^C(L) = d_T g$ is equivalent to $L(Z^C)\theta_L^A = d\tau^*g^A$, and $Z^C(E_L) = 0$, which implies

$$L(Z^C)\omega_L^A = 0 \quad \text{and} \quad L(Z^C)E_L = 0 ,$$

that is, Z^C is an infinitesimal natural Cartan symmetry. Then by Theorem 3, $f = (f^1, \dots, f^k)$ is a conservation law. \blacksquare

In the case $k = 1$, this statement can be found in [6] and [34].

Finally, we also have that:

Theorem 4 (Noether): *If $Y \in \mathfrak{X}(T_k^1 Q)$ is an infinitesimal Cartan symmetry of a k -symplectic Lagrangian system $(T_k^1 Q, \omega_L^A, E_L)$ then, for every $\Gamma = (\Gamma_1, \dots, \Gamma_k) \in \mathfrak{X}_L^k(T_k^1 Q)$, we have*

$$L(\Gamma_A)f^A = 0 \quad (\text{on } U_p) .$$

(*Proof*) This is the same as for Theorem 2. \blacksquare

3.4 Equivalent Lagrangians

Given a k -symplectic Lagrangian system $(T_k^1 Q, \omega_L^A, E_L)$, we know that canonical lifting of diffeomorphisms and vector fields preserve the canonical structures of $T_k^1 Q$. Nevertheless, the k -symplectic structure given by the forms ω_L^A is not canonical, since it depends on the choice of the Lagrangian function L , and then it is not invariant by these canonical liftings. Thus, given a diffeomorphism $\Phi: T_k^1 Q \rightarrow T_k^1 Q$ or a vector field $Y \in \mathfrak{X}(T_k^1 Q)$, a sufficient condition to assure the conditions (a) and (b) in definition 10 would be to demand that Φ or Y leave the canonical endomorphisms S^A and the Liouville vector field Δ invariant (for instance, Φ and Y being the canonical lifting of a diffeomorphism and a vector field in Q), and that the Lagrangian function L be also invariant. In this way, ω_L^A , E_L and hence the Euler-Lagrange equations are invariant by Φ or Y . However, to demand the invariance of L is a strong condition, since there are Lagrangian functions that, being different, give rise to the same k -symplectic structure ω_L^A , $A = 1, \dots, k$, and the same Euler-Lagrange equations. Thus, following the same terminology as in mechanics (see [1]), we can define:

Definition 11 *Two Lagrangian functions $L_1, L_2 \in C^\infty(T_k^1 Q)$ are gauge equivalent if*

1. $\omega_{L_1}^A = \omega_{L_2}^A$, for $A = 1, \dots, k$.
2. $\mathfrak{X}_{L_1}^k(T_k^1 Q) = \mathfrak{X}_{L_2}^k(T_k^1 Q)$.

Gauge equivalent Lagrangians can be also characterized as follows:

Proposition 12 *Two Lagrangians $L_1, L_2 \in C^\infty(T_k^1 Q)$ are gauge equivalent if, and only if,*

1. $\omega_{L_1}^A = \omega_{L_2}^A$, for $A = 1, \dots, k$.
2. $E_{L_1} = E_{L_2}$, (up to a constant).

(Proof) We will prove that, if $\omega_{L_1}^A = \omega_{L_2}^A$, for $A = 1, \dots, k$, then $\mathfrak{X}_{L_2}^k(T_k^1 Q) = \mathfrak{X}_{L_1}^k(T_k^1 Q)$ is equivalent to $E_{L_1} = E_{L_2}$ (up to a constant).

If $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_{L_2}^k(T_k^1 Q) = \mathfrak{X}_{L_1}^k(T_k^1 Q)$, then

$$0 = \sum_{A=1}^k i(X_A)\omega_{L_1}^A - dE_{L_1} = \sum_{A=1}^k i(X_A)\omega_{L_2}^A - dE_{L_2}$$

but as $\omega_{L_1}^A = \omega_{L_2}^A$, this implies that $dE_{L_1} = dE_{L_2}$, and hence $E_{L_1} = E_{L_2}$, up to a constant.

Conversely, if $\omega_{L_1}^A = \omega_{L_2}^A$, and $E_{L_1} = E_{L_2}$ (up to a constant), then for every $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_{L_1}^k(T_k^1 Q)$, we have

$$0 = \sum_{A=1}^k i(X_A)\omega_{L_1}^A - dE_{L_1} = \sum_{A=1}^k i(X_A)\omega_{L_2}^A - dE_{L_2}$$

so $\mathbf{X} \in \mathfrak{X}_{L_2}^k(T_k^1 Q)$, and in the same way we prove that if $\mathbf{X} \in \mathfrak{X}_{L_2}^k(T_k^1 Q)$, then $\mathbf{X} \in \mathfrak{X}_{L_1}^k(T_k^1 Q)$. ■

For gauge-equivalent Lagrangians, definition 11 guarantees the invariance of the set of k -vector fields which are solution to the geometric Euler-Lagrange equations (28). Nevertheless, this condition is also sufficient to assure the invariance of the set solutions to the Euler-Lagrange equations (22). In fact:

Proposition 13 *If the Lagrangian functions $L_1, L_2 \in C^\infty(T_k^1 Q)$ are gauge equivalent then, the Euler-Lagrange equations (22) associated to L_1 and L_2 have the same solutions.*

(Proof) If $L_1, L_2 \in C^\infty(T_k^1 Q)$ are gauge equivalent, then by the Proposition 12 we have: $\omega_{L_1}^A = \omega_{L_2}^A$, for $A = 1, \dots, k$ and $E_{L_1} = E_{L_2}$, (up to a constant). As $\omega_{L_1}^A = \omega_{L_2}^A$, for $A = 1, \dots, k$, from (25) we deduce that

$$\frac{\partial^2 L_1}{\partial q^j \partial v_A^i} = \frac{\partial^2 L_2}{\partial q^j \partial v_A^i} \quad \text{and} \quad \frac{\partial^2 L_1}{\partial v_B^j \partial v_A^i} = \frac{\partial^2 L_2}{\partial v_B^j \partial v_A^i}. \quad (46)$$

Therefore, we obtain

$$\begin{aligned} & \frac{\partial}{\partial t^A} \left(\frac{\partial L_1}{\partial v_A^i} \Big|_{\phi^{(1)}(t)} \right) \frac{\partial^2 L_1}{\partial q^j \partial v_A^i} \Big|_{\phi^{(1)}(t)} \frac{\partial \phi^j}{\partial t^A} \Big|_t + \frac{\partial^2 L_1}{\partial v_B^j \partial v_A^i} \Big|_{\phi^{(1)}(t)} \frac{\partial^2 \phi^j}{\partial t^A \partial t^B} \Big|_t \\ &= \frac{\partial^2 L_2}{\partial q^j \partial v_A^i} \Big|_{\phi^{(1)}(t)} \frac{\partial \phi^j}{\partial t^A} \Big|_t + \frac{\partial^2 L_2}{\partial v_B^j \partial v_A^i} \Big|_{\phi^{(1)}(t)} \frac{\partial^2 \phi^j}{\partial t^A \partial t^B} \Big|_t = \frac{\partial}{\partial t^A} \left(\frac{\partial L_2}{\partial v_A^i} \Big|_{\phi^{(1)}(t)} \right). \end{aligned} \quad (47)$$

Furthermore, $E_{L_1} = E_{L_2}$ (up to a constant), then $\frac{\partial E_{L_1}}{\partial q^j} = \frac{\partial E_{L_2}}{\partial q^j}$, and from (26) we deduce

$$v_A^i \frac{\partial^2 L_1}{\partial q^j \partial v_A^i} - \frac{\partial L_1}{\partial q^j} = v_A^i \frac{\partial^2 L_2}{\partial q^j \partial v_A^i} - \frac{\partial L_2}{\partial q^j}. \quad (48)$$

From (46) and (48) we obtain

$$\frac{\partial L_1}{\partial q^j} = \frac{\partial L_2}{\partial q^j}, \quad (49)$$

and then, from (47) and (49) we obtain

$$\sum_{A=1}^k \frac{\partial}{\partial t^A} \left(\frac{\partial L_1}{\partial v_A^i} \Big|_{\phi^{(1)}(t)} \right) - \frac{\partial L_1}{\partial q^j} \Big|_{\phi^{(1)}(t)} = \sum_{A=1}^k \frac{\partial}{\partial t^A} \left(\frac{\partial L_2}{\partial v_A^i} \Big|_{\phi^{(1)}(t)} \right) - \frac{\partial L_2}{\partial q^j} \Big|_{\phi^{(1)}(t)},$$

which implies that $\phi: \mathbb{R}^k \rightarrow Q$ is a solution to the Euler-Lagrange equations associated to L_1 if, and only if, it is a solution to the Euler-Lagrange equations associated with L_2 . ■

As a generalization of an analogous result in mechanics (see [1], p 216), we have the following results:

Proposition 14 *A Lagrangian $L: T_k^1 Q \rightarrow \mathbb{R}$ satisfies $\omega_L^A = 0$, for every $A = 1, \dots, k$, if, and only if, there exist $\alpha^1, \dots, \alpha^k \in \Omega^1(Q)$, closed 1-forms on Q and a function $f \in C^\infty(Q)$, such that $L = \hat{\alpha} + \tau^* f$ (up to a constant), where $\hat{\alpha} \in C^\infty(T_k^1 Q)$ is the function defined by*

$$\begin{aligned} \hat{\alpha} : T_k^1 Q &\longrightarrow \mathbb{R} \\ w_q = (v_{1q}, \dots, v_{kq}) &\mapsto \sum_{A=1}^k \alpha_q^A(v_{Aq}) \end{aligned} \quad .$$

(Proof) Suppose that $\omega_L^A = -d\theta_L^A = 0$, $1 \leq A \leq k$, then $\theta_L^A = dL \circ S^A$ are closed and semi-basic 1-forms on $T_k^1 Q$, then $dL \circ S^A$ are basic forms and there exist $\alpha^A \in \Omega^1(Q)$ such that

$$dL \circ S^A = \tau^* \alpha^A, \quad 1 \leq A \leq k. \quad (50)$$

Moreover, since $0 = d\theta_L^A = d(\tau^* \alpha^A) = \tau^*(d\alpha^A)$, then $d\alpha^A = 0$; that is, each α^A is a closed 1-form on Q . Furthermore, by a computation in local coordinates we obtain $d\hat{\alpha} \circ S^A = \tau^* \alpha^A$, and from (50) we have $d\hat{\alpha} \circ S^A = \tau^* \alpha^A = dL \circ S^A$. Then $d(L - \hat{\alpha}) \circ S^A = 0$. Therefore, the 1-form $d(L - \hat{\alpha})$ is closed and semi-basic. As a consequence, $d(L - \hat{\alpha})$ is a basic 1-form; that is, there exist $f \in C^\infty(Q)$ such that $d(L - \hat{\alpha}) = \tau^* df = d(\tau^* f)$. Then $L = \hat{\alpha} + \tau^* f$ (up to a constant).

Conversely, let us suppose that $L = \hat{\alpha} + \tau^* f$ (up to a constant). For every $A = 1, \dots, k$ we have

$$\theta_L^A = dL \circ S^A = d(\hat{\alpha} + \tau^* f) \circ S^A = d\hat{\alpha} \circ S^A = \tau^* \alpha^A,$$

since $d\tau^* f$ vanishes on the vertical vector fields. As α^A is closed, $d\alpha^A = 0$ and we obtain

$$\omega_L^A = -d\theta_L^A = -d(\tau^* \alpha^A) = -\tau^*(d\alpha^A) = 0.$$

■

Proposition 15 *The Lagrangian functions $L_1, L_2 \in C^\infty(T_k^1 Q)$ are gauge equivalent if, and only if, $L_1 = L_2 + \hat{\alpha}$ (up to a constant).*

(Proof) Let us suppose that $L_1, L_2 \in C^\infty(T_k^1 Q)$ are gauge equivalent. As $\omega_{L_1}^A = \omega_{L_2}^A$, then $\omega_{L_1 - L_2}^A = 0$, $1 \leq A \leq k$. Thus, by Proposition 14, there exist $\alpha^1, \dots, \alpha^k \in Z^1(Q)$ and $f \in C^\infty(Q)$ such that $L_1 - L_2 = \hat{\alpha} + \tau^* f$ (up to a constant).

From Proposition 12 we know that $E_{L_1} = E_{L_2}$, (up to a constant), or equivalently, $E_{L_1} - E_{L_2} = 0$ (up to a constant). Therefore,

$$\begin{aligned} 0 &= E_{L_1} - E_{L_2} = \Delta(L_1) - L_1 - \Delta(L_2) + L_2 = \Delta(L_1 - L_2) - (L_1 - L_2) \\ &= \Delta(\hat{\alpha} + \tau^* f) - (L_1 - L_2) = \hat{\alpha} - (L_1 - L_2) \quad (\text{up to a constant}). \end{aligned}$$

Conversely, let us suppose $L_1 = L_2 + \hat{\alpha}$ (up to a constant). First, a simple computation gives

$$\begin{aligned} \omega_{L_2}^A - \omega_{L_1}^A &= d(\theta_{L_1}^A - \theta_{L_2}^A) = d(d(L_1 - L_2) \circ S^A) = d(d\hat{\alpha} \circ S^A) = d(\tau^* \alpha^A) \\ &= \tau^*(d\alpha^A) = 0. \end{aligned}$$

Thus $\omega_{L_1}^A = \omega_{L_2}^A$. Furthermore,

$$E_{L_1} = \Delta(L_1) - L_1 = \Delta(L_2 + \hat{\alpha}) - (L_2 + \hat{\alpha}) = E_{L_2} + \hat{\alpha} - \hat{\alpha} = E_{L_2} \quad (\text{up to a constant}),$$

since $\Delta(\hat{\alpha}) = \hat{\alpha}$. As $\omega_{L_1}^A = \omega_{L_2}^A$ and $E_{L_1} = E_{L_2}$ (up to a constant), which means that L_1 and L_2 are gauge equivalents (see Proposition 12). \blacksquare

3.5 Lagrangian gauge symmetries

Bearing in mind the discussion made in the last section, we can define:

Definition 12 *Let $(T_k^1 Q, \omega_L^A, E_L)$ be a k -symplectic Lagrangian system.*

1. *A Lagrangian gauge symmetry is a diffeomorphism $\Phi: T_k^1 Q \rightarrow T_k^1 Q$ such that L and $\Phi^* L$ are gauge-equivalent Lagrangians; that is, $\Phi^* L = L + \hat{\alpha}$ (up to a constant), $\hat{\alpha} \in C^\infty(T_k^1 Q)$ being the function defined in Proposition 14.*

In the particular case where $\Phi^ L = L$ (up to a constant), then Φ is said to be a Lagrangian strict symmetry.*

A Lagrangian gauge symmetry is said to be natural if there exists a diffeomorphism $\varphi: Q \rightarrow Q$ such that $\Phi = (T_k^1)\varphi$.

2. *An infinitesimal Lagrangian gauge symmetry is a vector field $Y \in \mathfrak{X}(TQ)$ whose local flows are Lagrangian gauge symmetries.*

In the particular case where $L(Y)L = 0$, then Y is said to be an infinitesimal Lagrangian strict symmetry.

An infinitesimal Lagrangian gauge symmetry is said to be natural if there exists a vector field $Z \in \mathfrak{X}(Q)$ such that $Y = Z^C$,

Remark: A Lagrangian gauge symmetry $\Phi: T_k^1 Q \rightarrow T_k^1 Q$ of a k -symplectic Lagrangian system is not necessarily a Cartan symmetry, since in general $\Phi^* \omega_L^A \neq \omega_{\Phi^* L}^A$, for $A = 1, \dots, k$, and $\Phi^* E_L \neq E_{\Phi^* L}$, as can be easily proved with a simple calculation in coordinates.

In general we have:

Lemma 4 *Let $\varphi: Q \rightarrow Q$ be a diffeomorphism and let $\Phi = T_k^1(\varphi)$ the canonical prolongation of φ . Then:*

$$(i) \Phi^* \theta_L^A = \theta_{\Phi^* L}^A, \quad (ii) \Phi^* \omega_L^A = \omega_{\Phi^* L}^A, \quad (iii) \Phi^* E_L = E_{\Phi^* L}.$$

(Proof) This is a direct consequence of Lemma 2 and the definition of θ_L^A . In fact, for $\Phi = T_k^1(\varphi)$ we obtain

$$\begin{aligned} \Phi^* \theta_L^A &= \Phi^*(dL \circ S_A) = d(\Phi^* L) \circ S_A = \theta_{\Phi^* L}^A. \\ \Phi^* \omega_L^A &= \Phi^*(-d\theta_L^A) = -d\Phi^* \theta_L^A = \omega_{\Phi^* L}^A. \\ \Phi^* E_L &= \Phi^*\left(\sum_{A=1}^k \Delta_A(L) - L\right) = \sum_{A=1}^k \Delta_A(\Phi^* L) - \Phi^* L = E_{\Phi^* L}. \end{aligned}$$

\blacksquare

And then we have the following relation between natural Cartan symmetries and natural gauge symmetries:

Proposition 16 *Let $(T_k^1 Q, \omega_L^A, E_L)$ be a k -symplectic Lagrangian system. Then, $\Phi: T_k^1 Q \rightarrow T_k^1 Q$ is a natural Cartan symmetry if, and only if, it is a natural Lagrangian gauge symmetry.*

(Proof) If $\Phi = T_k^1(\varphi)$ for some diffeomorphism $\varphi: Q \rightarrow Q$, by lemma (4) we have that

$$\Phi^* \omega_L^A = \omega_{\Phi^* L}^A \quad , \quad \Phi^* E_L = E_{\Phi^* L}$$

therefore

$$\left. \begin{array}{l} \Phi^* \omega_L^A = \omega_{\Phi^* L}^A \\ \Phi^* E_L = E_{\Phi^* L} \end{array} \right\} \iff \left\{ \begin{array}{l} (\omega_{\Phi^* L})_A = \omega_L^A \\ E_{\Phi^* L} = E_L \end{array} \right.$$

that is, Φ is a natural Cartan Lagrangian symmetry if, and only if, L and $\Phi^* L$ are gauge equivalent Lagrangians and thus Φ is a natural Lagrangian gauge symmetry. ■

This result also holds for infinitesimal Lagrangian symmetries, taking the corresponding local flows.

Finally, we can state a particular version of Noether's theorem for natural Lagrangian strict symmetries:

Theorem 5 (Lagrangian Noether): *If $Y \in \mathfrak{X}((T_k^1 Q))$ is an infinitesimal natural Lagrangian strict symmetry of a k -symplectic Lagrangian system $(T_k^1 Q, \omega_L^A, E_L)$, with $Y = Z^C$, for some $Z \in \mathfrak{X}(T_k^1 Q)$, then the functions $f^A = Z^{V_A}(L)$, for $1 \leq A \leq k$, define a conservation law $f = (f^1, \dots, f^k)$.*

(Proof) This is a straightforward consequence of the above proposition and corollary 1 since in this case,

$$d\zeta^A = L(Y)\theta_L^A = L(Z^C)\theta_L^A = 0 \quad , \quad 1 \leq A \leq k \quad .$$

■

In the case $k = 1$, the above result can be found in [2, 34].

4 Conclusions and outlook

We analyze several kinds of symmetries that can be defined for Hamiltonian and Lagrangian first-order classical field theories, in their k -symplectic formulation.

First, we define the concept of symmetry (and infinitesimal symmetry). Second, according to Olver, we define conservation laws and investigate the problem of associating conservation laws with symmetries. In this way we have considered Cartan symmetries (which preserve the k -symplectic structures and physics; i.e., the Hamiltonian or the energy function) and, in particular, those called “natural”, which are canonical liftings of diffeomorphisms or vector fields. We prove that Cartan symmetries are symmetries and that there is a natural way of associating them with conservation laws by means of Noether's theorem. We state and prove this theorem in different situations for the Hamiltonian and Lagrangian cases.

Finally, we study and characterize gauge equivalent Lagrangians, leading to the introduction of Lagrangian gauge symmetries (which transform a Lagrangian into another equivalent one), proving that natural Lagrangian gauge symmetries are the same as natural Cartan symmetries, and stating the corresponding Noether's theorem.

Further research will be devoted to extending all these concepts and results to the k -cosymplectic formalism of first-order classical field theories.

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